United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 1 : Thursday, 3 December 2009

Time allowed $3\frac{1}{2}$ hours.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.*
- *Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

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1. Find all integers x, y and z such that
$$x^2 + y^2 + z^2 = 2(yz + 1) \text{ and } x + y + z = 4018.$$

2. Points A, B, C, D and E lie, in that order, on a circle and the lines AB and ED are parallel. Prove that $\angle ABC = 90^\circ$ if, and only if, $AC^2 = BD^2 + CE^2$.

3. Isaac attempts all six questions on an Olympiad paper in order. Each question is marked on a scale from 0 to 10. He never scores more in a later question than in any earlier question. How many different possible sequences of six marks can he achieve?

4. Two circles, of different radius, with centres at B and C, touch externally at A. A common tangent, not through A, touches the first circle at D and the second at E. The line through A which is perpendicular to DE and the perpendicular bisector of BC meet at F. Prove that $BC = 2AF$.

5. Find all functions f, defined on the real numbers and taking real values, which satisfy the equation $f(x)f(y) = f(x + y) + xy$ for all real numbers x and y.

6. Long John Silverman has captured a treasure map from Adam McBones. Adam has buried the treasure at the point (x, y) with integer co-ordinates (not necessarily positive). He has indicated on the map the values of $x^2 + y$ and $x + y^2$, and these numbers are distinct. Prove that Long John has to dig only in one place to find the treasure.